Int J Theor Phys (2010) 49: 681–692
DOI 10.1007/s10773-010-0247-z

Holonomy Groups Coming from F-Theory Compactification

Adil Belhaj · Luis J. Boya · Antonio Seguí

Received: 11 November 2009 / Accepted: 8 January 2010 / Published online: 22 January 2010

Abstract We study holonomy groups coming from F-theory compactifications. We focus mainly on $SO(8)$ as $12 - 4 = 8$ and subgroups $SU(4)$, $Spin(7)$, G_2 and $SU(3)$ suitable for descent from F-theory, M-theory and Superstring theories. We consider the relation of these groups with the octonions, which is striking and reinforces their role in higher dimensions and dualities. These holonomy groups are related in various mathematical forms, which we exhibit.

Keywords Superstrings · M-theory · F-theory · Octonions · Holonomy groups

1 Why Extra Dimensions

Once one includes strings and/or other extended objects, extra dimensions became unavoidable: for example, particles dualize in four dimensions, like electrons and magnetic monopoles, but strings dualize in six, and membranes in eight. On the other hand, e.g. string fields like to share supersymmetric partners, so Susy also is more cogent in higher dimensions. Hence the necessity of dimensional reduction, introducing tiny compact spaces, as we see only four extended dimensions [1].

A. Belhaj
Centre National de l'Energie, des Sciences et des Techniques Nucléaires, Rabat, Morocco

A. Belhaj · L.J. Boya (✉) · A. Seguí
Departamento de Física Teórica, Universidad de Zaragoza, 50009 Zaragoza, Spain
e-mail: luisjo@unizar.es

A. Belhaj
e-mail: belhaj@unizar.es

A. Seguí
e-mail: segui@unizar.es

A. Belhaj
Groupement National de Physique des Hautes Energies, Siège focal: FSR, Rabat, Morocco

In this paper we discuss the geometry of these compact spaces from the point of view of holonomy groups and string dualities; we focus mainly on the mathematical side of the new constructions. The preferred extra freedoms are needed for a total of ten, eleven or twelve dimensions: superstrings live in ten (whereas the bosonic string needs 26), but M-Theory (Witten, 1995) prefers 11 [2–10], and F-Theory (Vafa, 1996, 2008) uses 12 [11–17].

Over the past few years, there has been an increasing interest in studying duality in relation to supersymmetry and compactified manifolds [1, 18–23]; by duality we first understand the naive concept that a field strength F of dimension d in a manifold of dimension D dualizes to another field strength F^* of dimension $D - d$. For example, strings couple to potential 2-forms $B_{\mu\nu}$, hence to a 3d field strength, and in $D = 10$ dimensions the dual of a fundamental string is a 5-brane. More geometric is the duality between some Calabi-Yau (CY) spaces, discovered by Candelas et al. in their search for interesting manifolds suitable for heterotic string compactification, called mirror symmetry [24]. The most important consequence of the string duality was found by Witten: namely, the five viable superstring theories are special limits in the moduli space of the same theory, called M-Theory [2]. M-Theory was considered for about ten years the best candidate for the unification of microscopic forces and also with gravity, as the low energy limit of this theory describes the well-known 11-dim supergravity [25]. However, M-Theory never illuminated anyone of the genuine features of the microphysical world: neither gauge groups, nor particle spectrum nor even the number of distinct forces were selected by M-Theory. So lately C. Vafa has resurrected a theory of his, of 1996 (F-Theory, which lives in 12 dimensions [11]) with the assumption that perhaps decoupling gravity the new F-Theory, in adequate compactification, can account for some of the features of the standard model [14, 15]. In this paper we consider the geometric peculiarities of these dimension differences, for objects living in 10, 11 or 12 dimensions.

Within strings supersymmetry is mandatory, lest we want to contemplate 26 dimensions; but we can only tolerate $\mathcal{N} = 1$ Susy in our mundane, $4d$ space. Parity violation is a conspicuous feature of our world, but with $\mathcal{N} > 1$ Susy, chiral partners are in the same multiplet; so parity is conserved. The preservation of this feature puts stringent conditions on the compactifying manifold with $10 - 4 = 6$ dimensions, and corresponding results obtains from descent $11 \to 4$ (M-Theory) and $12 \to 4$ (F-Theory) [11, 14, 15].

In particular, we need manifolds with $SU(3)$ holonomy groups for the heterotic string case [22], essentially because $4 = 3 + 1$ in the descent $Spin(6) \cong SU(4)$ to $SU(3)$. The possible manifolds are signalled CY_3. In M-Theory the chain is: descent $Spin(7)$ to G_2, because $8 = 7 + 1$ again. The restrictions on F-Theory depends on the signature (one or two times) and will be dealt with later. The further break from the gauge group (e.g. $E_8 \times E_8$ in the case of heterotic exceptional superstring) to more realistic Standard Model groups, like $E_8 \times E_6$, and further $E_6 \to SU(5) \to SU(3)_C \times SU(2)_L \times U(1)_Y$ requires additional mechanisms, of course, see e.g. [23]. More recently, the descent from 12 to 4 in F-Theory is managed in two or three steps, see below.

As a whole, there are many ways to get four-dimensional models using different compactifications as intermediates. There are relations between several of these constructs due to special dualities which appear in the process. As an example, we point out here different equivalences in seven dimensions giving rise to a web of dualities (with F and M for F-theory resp. M-Theory) [1]

$$F/K3 \times S^1 \sim het/T^3 \sim IIB/S^2 \times S^1 \sim M/K3 \sim IIA/S^2 \times S^1. \quad (1)$$

This can be pursued, of course, to lower dimensions. The main focus of this paper is the study of these compactifications down to four dimensions within the perspective of dualities.

We shall focus mainly in four groups, $SU(4)$, $Spin(7)$, G_2 and $SU(3)$. All these can be seen as subgroups of $SO(8)$, the maximal compact group of the F-Theory compactification down to four dimensions ($12 \to 4$): This can be obtained by breaking the space-time symmetry $SO(1,11)$ down to the subgroup $SO(1,3) \times SO(8)$. We shall not be much concerned with the manifolds themselves.

On the other hand, the relation of the above groups with the octonion division algebra should be evident [26], as e.g. $Spin(7)$ acts in the 7-sphere of unit octonions, G_2 in the 6-sphere of unit imaginary octonions, and $SU(3)$ on the equator of the later; we devote some space in Appendix to study in detail such connections. We shall exhibit also in Sect. 4 some exact triple sequences, which relate the precise mathematical relations between these holonomy groups.

The organization of the paper is as follows. In Sect. 2 we recall the classification of special holonomy manifolds by Berger (1955). In Sect. 3 we review different ways of constructing four-dimensional models with minimal number of supercharges from higher dimensional supersymmetric theories. Section 4 deals with F-theory and its relations with holonomy groups, and exhibits also the exact sequences mentioned above. The Appendix elaborates on generalities over the octonion division algebra.

2 Manifolds with Special Holonomy

2.1 About Holonomy Groups

The study of the supersymmetric theories and extended objects involves the prediction of extra dimensions of space-time. However, as we see only $4 = (1 + 3)$ dimensions, some mechanism has to be advocated to prevent the extra size of the space to be visible. The compactification is the most accepted ingredient, namely making the extra dimensions too small to be observable. In the original Kaluza-Klein type of theories (ca. 1920), the observable gauge group in 4 dimensions came from the *isometry* groups of the compactifying space (this is why the $U(1)$ gauge group of electromagnetism came from the compactification of the fifth dimension on a circle). But when supersymmetry is present, it was realized in the early 80s that the *holonomy* groups of intermediate spaces respond of the number of supercharges surviving in four dimensions [27].

In this section we review briefly the classification of special holonomy groups and manifolds in a form suitable for all later physical consideration. Note that the books of Joyce [28, 29] are the best modern references for this subject. Let $\mathcal{M}$ be any n dimensional differentiable manifold. The structure group of the tangent bundle is a subgroup of the general linear group, $GL(n,R)$. Now the maximal compact group of the linear group is $O(n)$. So the quotient homogeneous space $\frac{GL(n,R)}{O(n)}$ is a contractible space; hence, a manifold admits always a Riemannian metric g, whose tangent structure group is (a subgroup of) the orthogonal group. The isometry group $Isom(\mathcal{M})$ is the set of diffeomorphisms σ leaving g invariant. For spheres we have $Isom(S^n) = O(n+1)$; for torii $Isom(T^n) = U(1)^n$.

For an arbitrary n dimensional Riemannian manifold $\mathcal{M}$, the structure group of the tangent bundle is, as said, a subgroup of $O(n)$. Carrying a orthonormal frame ϵ of n vectors in a point P through a closed loop γ in the manifold, by parallel transport back to P,

$$\gamma : P \to P' \to P \tag{2}$$

it becomes another frame $\epsilon' = o \cdot \epsilon$ which is shifted by certain element o of $O(n)$. This is the *holonomy element* of the loop. All elements of all possible loops on the manifold from P to

P make up the holonomy group of the manifold $Hol_P(\mathcal{M})$, which is always a subgroup of $O(n)$, and is easily seen to be independent of the point P for an arcwise-connected manifold. A generic Riemannian manifold would have holonomy $O(n)$, or $SO(n)$ if it is orientable, whereas the isometry group is just the identity generically; in a way isometry and holonomy are complementary.

For any vector bundle with a connection, the structure group reduces to the holonomy group (reduction theorem). The corresponding Lie algebra of the holonomy group is generated by the curvature of the connection (the Ambrose-Singer theorem) [30].

2.2 Types of Special Holonomy Manifolds

The classification of special holonomy groups was carried by M. Berger in 1955. If the manifold is irreducible, $Hol(\mathcal{M})$ should lie in $O(n)$. Its Lie algebra, as we said, is generated by the curvature. For the irreducible non symmetric case, there are three double series of classes of manifolds corresponding to the numbers **R**, **C** and **H**, and two isolated cases related to the octonion numbers **O**. For each number domain there are the generic case and the unimodular subgroup restriction. The list practically coincides with the list of groups with transitive action on spheres.

The classification of holonomy groups is given in Table 1 [13].

Some explanations are in order. An arbitrary n-dimensional Riemannian manifold $\mathcal{M}$ has $O(n)$ as the maximal holonomy group. The obstruction to orientability is measured by the first Stiefel-Witney class of the tangent bundle, $w_1(\mathcal{M}) \equiv w_1(T\mathcal{M}) \in H^1(\mathcal{M}, Z_2)$.

A n-dimensional complex Kähler manifold parameterized by $z_i, i = 1, \dots, n$ has a closed regular real Kähler two-form ω given in a local chart by

$$\omega = i w_{i\bar{j}} dz_i \wedge d\bar{z}_{\bar{j}}, \quad d\omega = 0, \tag{3}$$

where $w_{i\bar{j}}$ can be expressed as

$$w_{i\bar{j}} = \frac{\partial^2 K}{\partial z_i \partial \bar{z}_{\bar{j}}}, \tag{4}$$

Table 1 Holonomy Groups

Numbers	Group	Unimodular form
R	$O(n)$	$SO(n)$
	generic case	orientable, $w_1 = 0$
C	$U(n)$	$SU(n)$
	Kähler, $d\omega = 0$	Calabi-Yau, $c_1 = 0$
H	$Sp(1) \times /_2 Sp(n)$	$Sp(n)$
	Quaternionic	Hyperkähler
O	$Spin(7)$ in $8d$ spaces	G_2 in $7d$ spaces
	$Oct(1)$	$Aut(\mathbf{O})$

and where K is a locally defined function called the Kähler potential. The holonomy group of such a geometry is $U(n)$. Now as $\frac{U(n)}{SU(n)} = U(1)$, we have the diagram

$$\begin{array}{l} SU(n) \\ \quad\downarrow \\ U(n) \longrightarrow B \longrightarrow M \\ {}_{\det}\!\downarrow \\ U(1) \longrightarrow B' \longrightarrow M \end{array} \tag{5}$$

where the middle line is the frame bundle: B is the principal bundle of unitary frames. The last bundle is mapped to an element of $H^2(\mathcal{M}, Z)$; hence, the determinant map defines the first Chern class of the bundle as $c_1(\mathcal{M}) \in H^2(\mathcal{M}, Z)$. It turns out then that when $c_1 = 0$, the Kähler manifold becomes a CY manifold with $SU(n)$ holonomy group and it is Ricci flat, $Ric = 0$; this is because ([28], p. 98), the Ricci tensor is equivalent to a 2-form proportional to c_1, which is zero in the CY case. Note that an one-dimensional CY manifold is nothing but a torus T^2, as $SU(1) = 1$. So its Hodge diamond is given by

$$\begin{array}{ccc} & h^{0,0} & \\ h^{1,0} & & h^{0,1} \\ & h^{1,1} & \end{array} = \begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array} \tag{6}$$

The second example of CY geometries is the K3 complex surface with $SU(2)$ as holonomy group. Its Hodge diamond reads

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1. \\ & 0 & & 0 & \\ & & 1 & & \end{array} \tag{7}$$

Since $SU(2) = Sp(1)$, the K3 surface is also a hyperkähler manifold: Notice that hyperkähler manifolds are also Calabi-Yau, but the quaternionic manifolds in general are not. Quaternionic manifolds have for holonomy $Sp(1) \times Sp(n)/Z_2$, abbreviated as $Sp(1) \times /_2 Sp(n)$ in Table 1.

Finally, the two cases related to the octonions are G_2, that is the octonion automorphism group and $Spin(7)$. The former is well known and we shall elaborate on it later; as for the "$Oct(1)$" label for $Spin(7)$, this will also be clarified in the Appendix.

Note that, in general, a manifold with a specific holonomy group $Hol(\mathcal{M}) = G$ implies the manifold carries an additional structure, preserved by the group G. For example, an orientable manifold, with holonomy within $SO(n)$, has an invariant volume element, indeed $SO(n) = O(n) \cap SL(n, R)$. A Kähler manifold, with holonomy inside $U(n)$ has an invariant closed 2-form as $U(n) = O(2n) \cap Sp(n)$; a $SU(n)$ holonomy manifold carries a holomophic volume. A manifold with G_2 holonomy will carry an invariant 3-form, etc.

3 Physical Compactification Scenarios

As we know superstring theory lives in ten dimensions [18]; down to four dimensions we want only $\mathcal{N} = 1$, i.e. four supercharges, as to allow for parity violation. We know that compactification on a $SU(n)$-holonomy manifold would reduce the supercharges by a factor of

Table 2 Diverse Compactifications

Theory	Dim change	Holonomy
Heter. string	$10d \longrightarrow 4d$	$SU(3)$ (CY_3 manifold, Ricci flat)
M-theory	$11d \longrightarrow 4d$	G_2 (Ricci flat)
M-theory	$11d \longrightarrow 3d$	$Spin(7)$ (Ricci flat)
F-theory (1, 11)	$12d \longrightarrow 4d$	$Spin(7), SU(4)$ (CY_4)
F-theory (2, 10)	$12d \longrightarrow 4d$	Indefinite form of $Spin(7)$ or $Spin(8)$

$1/2^{n-1}$, so $SU(3)$-holonomy (i.e., a CY_3) would be just right to descend from the heterotic string (16 supercharges) to a four dimensional model with only four supercharges. Indeed, the search for CY_3 manifolds was a big industry in the 80s [22]. This choice is also natural, as $SU(3) \subset SU(4) = Spin(6) \to SO(6)$, and obviously as $4 = 3 + 1$, $SU(3)$ leaves one surviving spinor.

In M-Theory living 11 dimensions, the candidate compactifying manifold would be one with G_2 holonomy group [4]: now the inclusions are $G_2 \subset SO(7) \leftarrow Spin(7)$, and $8 = 1 + 7$, as $2^{(7-1)/2} = 8$, type real. G_2-holonomy manifolds which are also Ricci flat, see again (Joyce, [28], p. 244) and would preserve $1/2^3$ supercharges, and in 11d there are $2^{(11-1)/2} = 32$, type real again as $10 - 1 = 9 \equiv 1 \bmod 8$.

We can also consider eight-dimensional compactifying manifolds in at least two contexts: 1) Descend $11 \to 3$ just for illustrative purposes, and 2) F-Theory with metric $(1, 11)$; the original suggestion of Vafa was $12 = (2, 10)$, see [11, 12]. Here the manifolds of choice would be either CY_4, that is, $SU(4)$-holonomy manifolds, preserving 4 supercharges out of 32 (which is what we want), or $Spin(7)$, the last of the exceptional holonomy groups; $Spin(7)$ does the job as it has an irreducible 8d representation, same as $Spin(8)$ and $Spin(7) \subset Spin(8)$. Table 2 sums up the situation.

We note that if we consider the conventional F-Theory with signature (2, 10) it is necessary to compactify in a manifold with signature $(1, 7)$.

4 Connections Between Holonomy Groups from F-theory

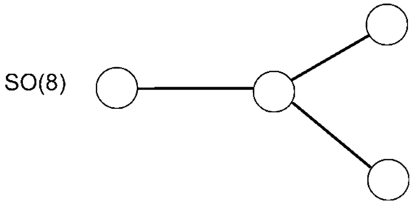
We consider now in the spirit of Vafa's new F-theory compactifications [14, 15] relations between special holonomy groups.

4.1 Holonomy Groups from F-theory Compactification

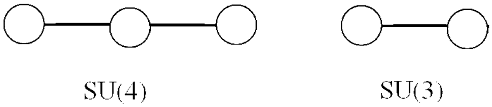
In F-theory compactification, with one time, the four dimensional models can be obtained by breaking the space-time symmetry $SO(1, 11)$ down to the following subgroup

$$SO(1, 11) \to SO(1, 3) \times SO(8) \tag{8}$$

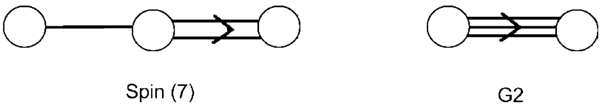
where $SO(8)$ is the maximal holonomy group of an eight-dimensional intermediate manifod X_8. All above discussed special holonomy group can be related to $SO(8)$ symmetry. As we know the Dynkin diagram for $SO(8)$ is given by



It is remarkable how the different special holonomy groups come from this Dynkin diagram. In particular, $SU(4)$ and $SU(3)$ can be obtained by deleting one and two nodes respectively



The Dynkin diagram of $SO(8)$ shows triality, the free permutations of the three external nodes. Identifying these nodes, we get the G_2 as the fix point set of S_3 group. The $Spin(7)$ group can be obtained by identifying the right spin nodes:



As we see, all special holonomy groups can be related to the maximal holonomy of the F-theory compactifications down to four dimensions.

4.2 Relations Between Different Special Holonomy Groups

Strings, M and F-theories are related by different sorts of dualities and compactifications. As a consequence we expect that also the different holonomy groups used in various compactifications should be connected. In what follows we address this question using exact sequences and commutative diagrams for these groups. To start, we note the following. If $H \subset G$ with (left)-coset space X, we write $H \to G \to X$ for $G/H = X$; when H is normal, X becomes the quotient group. Some of next diagrams have been already given in [31].

The first diagram that we present here comes form the inclusion of the exceptional holonomy group $G_2 \subset Spin(7)$. The later acts transitively in all units in $\mathbf{O}$ (octonions of norm one, forming S^7), whereas $G_2 = Aut(\mathbf{O})$ obviously leaves 1 invariant (the real part of the octonion). So the main cross of the diagram takes the following form

$$\begin{array}{ccccc} & & Spin(6) & & \\ & & \downarrow & & \\ G_2 & \longrightarrow & Spin(7) & \longrightarrow & S^7 \\ & & \downarrow & & \\ & & S^6 & & \end{array} \tag{9}$$

where the vertical line is elemental.[1] With the $A_3 = D_3$ isomorphism $Spin(6) = SU(4)$ and the fact that $SU(3) \subset G_2 \cap (Spin(6) = SU(4))$, we can complete the previous cross. The result is given by the diagram

$$\begin{array}{ccccccc} SU(3) & \longrightarrow & SU(4) & = Spin(6) & \longrightarrow & S^7 & \\ \downarrow & & & \downarrow & & \| & \\ G_2 & \longrightarrow & & Spin(7) & \longrightarrow & S^7. & \\ \downarrow & & & \downarrow & & & \\ S^6 & === & & S^6 & & & \end{array} \tag{10}$$

From this picture we can see in particular the octonionic nature of $SU(3)$. It is a group of automorphism of octonions, fixing the product, say $(ij)k$. There is a suspicion, still conjectural, that this is the reason why the gauge group of the strong forces is $SU(3)$ color.
To get the second diagram, we start by another obvious cross, since $Spin(7)$ is the (universal) double cover of $SO(7)$. In this way, we have the following diagram

$$\begin{array}{ccccc} & & Z_2 & & \\ & & \downarrow & & \\ G_2 & \longrightarrow & Spin(7) & \longrightarrow & S^7. \\ & & \downarrow & & \\ & & SO(7) & & \end{array} \tag{11}$$

It is known that G_2 does not have a centre, so $Z_2 = Z_2$ must be the upper row. The rest is easy to complete since S^7/Z_2 is the real projective space RP^7. We end up with the following picture

$$\begin{array}{ccccc} & & Z_2 & === & Z_2 \\ & & \downarrow & & \downarrow \\ G_2 & \longrightarrow & Spin(7) & \longrightarrow & S^7. \\ \| & & \downarrow & & \downarrow \\ G_2 & \longrightarrow & SO(7) & \longrightarrow & RP^7 \end{array} \tag{12}$$

From this diagram one can learn just the lower row, somewhat unexpected, until one sees the middle row. The lower row is also a remainder that the orthogonal groups have torsion [32].

In what follows, we incorporate the $SO(8)$ symmetry of F-theory compactifications in the diagrams. We will give two diagrams connecting $SO(8)$ with special holonomy groups. In terms of CY holonomy groups, we have the following picture

$$\begin{array}{cccccc} SU(3) & \longrightarrow & SU(4) & = Spin(6) & \longrightarrow & S^7 \\ \downarrow & & & \downarrow & & \| \\ Spin(7) & \longrightarrow & & SO(8) & \longrightarrow & S^7, \\ \downarrow & & & \downarrow & & \\ X_{13} & === & & X_{13} & & \end{array} \tag{13}$$

where X_{13} is a 13-dimensional homogeneous space.

The last diagram is obtained by asking the question how does $Spin(7)$ act transitively and isometrically in the seventh sphere S^7. Indeed, it must be a subgroup of $SO(8)$. What about

[1] The spin groups acting on the natural spheres via the SO (covered) groups, $Spin(n)/Z_2 = SO(n)$.

the quotient (homogeneous space)? To answer this question, we start first with the following incomplete cross

$$\begin{array}{ccccc} & & Spin(7) & \longrightarrow & S^7 \\ & & \downarrow & & \| \\ SO(7) & \longrightarrow & SO(8) & \longrightarrow & S^7 \\ & & \downarrow & & \\ & & ?? & & \end{array} \tag{14}$$

and then try to finish it. Indeed, G_2 lies inside both $Spin(7)$ and $SO(7)$, then it must be their intersection and must appear in the upper left corner. The rest of the diagram can be obtained easily, and the result is

$$\begin{array}{ccccc} G_2 & \longrightarrow & Spin(7) & \longrightarrow & S^7 \\ \downarrow & & \downarrow & & \| \\ SO(7) & \longrightarrow & SO(8) & \longrightarrow & S^7. \\ \downarrow & & \downarrow & & \\ RP^7 & === & RP^7 & & \end{array} \tag{15}$$

The new result we learn is just the middle column involving th maximal holonomy group of F-theory compactification to four dimensions.

This completes our study of the relations between holonomy groups which are suitable for the compactification of superstrings, M, F-theories respectively. We have found three triple exact sequences explaining some links between these groups. One of the nice results that one gets from the diagrams is that one can also see the possible connections between the corresponding geometries. Indeed, from the following sequence of inclusions

$$SU(3) \longrightarrow G_2 \longrightarrow S^6, \tag{16}$$

one can see that the manifold with G_2 holonomy can be constructed in terms of CY three folds with the $SU(3)$ holonomy group [28]. The construction is given by the following seven dimensional orbifold space

$$X_7(G_2) = \frac{CY_3 \times S^1}{Z_2} \tag{17}$$

The Betti numbers of $X_7(G_2)$ can be fixed by the Hodge numbers of CY_3, which are given by the number of their two and three non trivial cycles. Z_2 acts as the reverse transformation ($x \to -x$) on the circle S^1 and as an involution in the CY_3 space in order to mantain the G_2 structure. The action on the CY_3 is just a simply complex conjugation of its complex coordinates, ($z_i \to \bar{z}_i$). In this way, the associative 3-form Ψ of $X_7(G_2)$ can be expressed as

$$\Psi = \omega \wedge dx + Re(\Omega). \tag{18}$$

It is then easy to see that the G_2 structure is preserved by the involution, since both the Kahler form ω of the CY_3 and the one-form dx of the circle change sign while the holomophic $Re(\Omega = dz_1 \wedge dz_2 \wedge dz_3)$ is invariant. We can suppose the same thing for the manifold with $Spin(7)$ holonomy, it can be constructed either from manifold with G_2 holonomy or CY_4. This can be easily seen from the subdiagram (10).

Acknowledgements This work has been supported by CICYT (grant FPA-2006-02315) and DGIID-DGA (grant 2007-E24/2), Spain. We thank also the support by Física de altas energias: Partículas, Cuerdas y Cosmologia, A9335/07. AB would like to thank Departamento de Física Teórica of Zaragoza University for kind hospitality

Appendix: The Octonions

We recall here some properties of division algebras [33] in relation with special holonomy groups and manifolds. Starting with the real numbers **R**, the space R^2 becomes an algebra with $i \equiv \{0, 1\}$ and $i^2 = -1$: we get the complex number **C**. It is a commutative and associative division algebra. Adding a second unit j, $j^2 = -1$ a third ij is necessary, with $ij = -ji$, and we obtain the division algebra of quaternions **H** in R^4. It is anticommutative but still associative. Adding another independent unit k to i and j, with $k^2 = -1$, $ik = -ki$, $jk = -kj$, we have to complete with $e_7 = (ij)k$ to the algebra of octonions **O** in R^8, with units $1; i, j, k; ij, jk, ki; (ij)k = -i(jk)$. It is neither commutative nor associative, but still a division algebra: if $o = u_0 + \Sigma_{i=1}^7 u_i e_i$ we have

$$\bar{o} := u_0 - \Sigma_{i=1}^7 u_i e_i \qquad \mathcal{N}(o) = \text{norm}(o) := \bar{o}o; \quad \text{inverse}\, o^{-1} = \frac{\bar{o}}{\mathcal{N}(o)}. \tag{19}$$

The associator $\{o_1, o_2, o_3\} := (o_1 o_2)o_3 - o_1(o_2 o_3)$ is completely antisymmetric. The four algebras **R**, **C**, **H**, **O** are composition algebras, that is, we have $\mathcal{N}(xy) = \mathcal{N}(x)\mathcal{N}(y)$. The automorphism groups of the algebra are easily seen to be

$$Aut(\mathbf{R}) = 1, \qquad Aut(\mathbf{C}) = Z_2, \qquad Aut(\mathbf{H}) = SO(3), \qquad Aut(\mathbf{O}) := G_2. \tag{20}$$

The norm-one elements form, for **R**, **C**, **H**, **O**, respectively

$$\begin{aligned} &O(1) = Z_2 = S^0; \qquad U(1) = SO(2) = S^1; \\ &Sp(1) = SU(2) = Spin(3) = S^3; \quad \text{and} \quad S^7. \end{aligned} \tag{21}$$

Now S^7 has an invertible product structure, in particular is parallelizable, but is not a group, because nonassociativity. Let us name jokingly '$Oct(1)' = S^7$. One obtains a *bona fide* group by stabilizing S^7 by the octonion automorphism group G_2 [34]. The result is $Spin(7) \approx G_2(\times S^7$; we shall name $Spin(7) := Oct(1)$, where $(\times$ just means twisted product [35]. We recall now the description of compact Lie groups as finitely twisted products of odd dimensional spheres (Hopf 1941); for details see [35]. For example in the quaternion case one gets the sequence

$$\begin{aligned} &Sp(1) = Spin(3) = S^3, \qquad Sp(1)^2 = Spin(4) = S^3 \times S^3, \\ &Sp(2) = Spin(5) = S^3(\times S^7. \end{aligned} \tag{22}$$

There are analogous results for the octonions, after G_2 stabilization. The series goes up to dim 3, but not beyond; this is due to the lack of associativity. We just write the results, adding the sphere exponents

$$\begin{aligned} &G_2 = SOct(1) \approx S^3(\times S^{11}; \qquad Spin(8) = Oct(1)^2 \approx S^3(\times S^7(\times S^7(\times S^{11} \\ &Spin(9) := Oct(2) \approx S^3(\times S^7(\times S^{11}(\times S^{15}; \qquad F_4 := SOct(3) \approx S^3(\times S^{11}(\times S^{15}(\times S^{23} \end{aligned} \tag{23}$$

where by the prefix "S" we mean the unimodular restriction (no S^7 factors). This is similar to SO and SU for **R** and **C** respectively. The usefulness of the notation can be seen e.g. in the projective line and plane:

$$\begin{aligned} &HP^1 = S^4 = Sp(2)/Sp(1)^2 \quad \text{corresponds to} \quad OP^1 = S^8 = Spin(9)/Spin(8), \\ &CP^2 = S^5/S^1 = SU(3)/U(2) \quad \text{corresponds to} \quad OP^2 = SOct(3)/Oct(2) = F_4/Spin(9). \end{aligned} \tag{24}$$

The later is called the Moufang plane (Moufang 1933; to call it the Cayley plane is historically inaccurate).

In any case, this use, $G_2 \sim SOct(1)$ etc., is just a notational convention, that we find useful, if carefully employed. We finish by remarking that little use has been made so far of the fundamental *triality* property of the $O(8)$ group and the octonions, namely $Out[Spin(8)] \sim Aut/Inner = S_3$, the order three symmetric group. Perhaps in a deeper analysis this triality will show up in particle physics.

The necessary properties we need of the division algebra of the octonions **O** are described above. Here we recall that G_2 is the automorphism group of the octonions (as $SO(3)$ is $Aut(\mathbf{H})$ and $Z_2 = Aut(\mathbf{C})$); the reals **R** have not autos, hence the representation 8 of G_2 in the octonions split naturally in $8 = 1 + 7$. Note that G_2 acts transitively in the S^6 sphere of unit imaginary octonions. This implies the 6-sphere acquires a quasi-complex structure (Borel-Serre). The sequence reads as follows

$$SU(3) \to G_2 \to S^6 \quad (8 + 6 = 14) \tag{25}$$

where $SU(3)$ acts in the equator $S^5 \in R^6$ as the representation $\bar{3} + 3$. Now the octonionic product preserved by G_2, as any algebra $(xy = z)$, defines an invariant T_2^1 tensor and the conservation of the norm is like preserving a quadratic form. The T_2^1 tensor can be seen then as a T_3^0 tensor. Now the alternating property of the octonionic product is equivalent to this T_3^0 tensor to become a 3-form in R^7, $\wedge T_3^0$, which is generic, (i.e., they make up an open set). This implies

$$\dim G_2 = \dim GL(7, R) - \dim \wedge T_3^0 = 49 - 35 = 14. \tag{26}$$

Besides, the dual form $\wedge T_4^0$ is also invariant, implying G_2 is unimodular, i.e. lies inside $SO(7)$. The dimension 14 of this G_2 can of course be proved directly [36].

As with respect to $Spin(7)$, it has a real 8-dimensional representation as we said, and hence it acts in S^7, indeed transitively. The little group acts in the S^6 equator, and it is certainly G_2. In fact, there is some sense, as explained above, to call G_2 and $Spin(7)$ respectively $SOct(1)$ and $Oct(1)$.

References

1. Vafa, C.: Lectures on strings and dualities. hep-th/9702201
2. Witten, E.: String theory dynamics in various dimensions. Nucl. Phys. B **443**, 85 (1995). hep-th/9503124
3. Townsend, P.K.: The eleven-dimensional supermembrane revisited. Phys. Lett. B **350**, 184 (1995). hep-th/9501068
4. Papadopoulos, G., Townsend, P.: Compactification of $D = 11$ supergravity on spaces of exceptional holonomy. Phys. Lett. B **357**, 300–306 (1995)
5. Duff, M.: The World in Eleven Dimensions. Institute of Physics, Bristol (1999)
6. Acharya, B., Witten, E.: Chiral fermions from manifolds of G_2 holonomy. hep-th/0109152
7. Belhaj, A.: M-theory on G_2 manifolds and the method of (p, q) brane webs. J. Phys. A **37**, 5067–5081 (2004)
8. Belhaj, A.: F-theory duals of M-theory on G_2 manifolds from mirror symmetry. J. Phys. A **36**, 4191–4206 (2003). hep-th/0207208
9. Belhaj, A., Garcia del Moral, M.P., Restuccia, A., Segui, A., Veiro, J.P.: The supermembrane with central charges on a G2 manifold. J. Phys. A **42**, 325201 (2009). arXiv:0803.1827 [hep-th]
10. Belhaj, A., Drissi, L.B., Rasmussen, J.: On $N = 1$ gauge models from geometric engineering in M-theory. Class. Quantum. Gravity **20**, 4973–4982 (2003). hep-th/0304019
11. Vafa, C.: Evidence for F-theory. Nucl. Phys. B **469**, 403415 (1996). hep-th/9602022
12. Boya, L.J.: Arguments for F-theory. Mod. Phys. Lett. A **21**, 287–304 (2006)

13. Boya, L.J.: Special holonomy manifolds in physics. Rev. Acad. Cienc. Zaragoza **29**, 37–48 (2006). math-ph/0612002
14. Beasley, C., Heckman, J.J., Vafa, C.: GUTs and exceptional Branes in F-theory—I. arXiv:0802.3391 [hep-th]
15. Beasley, C., Heckman, J.J., Vafa, C.: GUTs and exceptional branes in F-theory—II: experimental predictions. arXiv:0806.0102 [hep-th]
16. Ahl Laamara, R., Belhaj, A., Boya, L.J., Segui, A.: On local F-theory geometries and intersecting D7-branes. arXiv:0902.1161 [hep-th]
17. Ahl Laamara, R., Belhaj, A., Boya, L.J., Medari, L., Segui, A.: On F-theory quiver models and kac-moody algebras. arXiv:0910.4852 [hep-th]
18. Green, M., Schwarz, J.H., Witten, E.: Superstring Theory, vols. 1, 2. Cambridge University Press, Cambridge (1986)
19. Sen, A.: Heterotic string theory and Calabi Yau manifolds in the Green-Schwarz formalism. Nucl. Phys. B **355**, 423 (1987)
20. Ooguri, H., Yin, Z.: TASI lectures on perturbative string theories. hep-th/9612254
21. Aspinwall, P.: K3 surfaces and string duality. hep-th/9611137
22. Candelas, P., et al.: Vacuum configurations for superstrings. Nucl. Phys. B **258**, 46 (1985)
23. Witten, E.: New issues in manifolds of $SU(3)$ holonomy. Nucl. Phys. B **268**, 79112 (1986)
24. Candelas, P., et al.: A pair of Calabi-Yau manifolds as an exact soluble superconformal theory. Nucl. Phys. B **359**, 21–74 (1991)
25. Cremmer, E., Julia, B., Scherk, J.: Supergravity theory in eleven dimensions. Phys. Lett. B **76**, 409–412 (1978)
26. Belhaj, A., Boya, L.J., Segui, A.: Relation between holonomy groups in superstrings, M and F-theories. arXiv:0806.4265 [hep-th]
27. Awada, M., et al.: Phys. Rev. Lett. **50**, 294–287 (1983)
28. Joyce, D.: Compact Manifolds with Special Holonomy. Oxford Univ. Press, London (2000)
29. Joyce, D.: Riemannian Holonomy Groups and Calibrated Geometry. Oxford Univ. Press, London (2007)
30. Kobayashi, S., Nomizu, K.: Foundations of Differential Geometry, vol. 1. Wiley, New York (1963). II.8.-J
31. Boya, L.J.: Spinors and octonions. math-ph/0409077
32. Borel, A.: Topology of Lie groups and characteristic classes. Bull. Am. Math. Soc. **61**, 397–411 (1955)
33. Baez, J.: The octonions. Bull. Am. Math. Soc. **39**, 145–205 (2002)
34. Ramond, R.: Introduction to exceptional Lie groups and algebras. Caltech Preprint CALT-68-577 (December 1976, unpublished)
35. Boya, L.J.: The geometry of compact Lie groups. Rep. Math. Phys. **30**, 149–162 (1991)
36. Rosenfeld, B.: Geometry of Lie Groups. Kluwer Academic, Dordrecht (1997), p. 57